Solution to Math4230 Tutorial 10

1. Consider MC/MC framework, assuming that w^* is finite, and $\bar{M}$ is convex and closed. Show that $q^* = w^*$.

   Solution:

   We prove this result by showing that all the assumptions of MC/MC Strong Duality (Prop. 4.3.1) are satisfied. By assumption, $w^* < \infty$ and the set $\bar{M}$ is convex. Therefore, we only need to show that for every sequence $\{u_k, w_k\} \subset M$ with $u_k \to 0$, there holds $w^* \leq \liminf_{k\to\infty} w_k$.

   Consider a sequence $\{u_k, w_k\} \subset M$ with $u_k \to 0$. If $\liminf_{k\to\infty} w_k = \infty$, then we are done, so assume that $\liminf_{k\to\infty} w_k = \tilde{w}$ for some scalar $\tilde{w}$. Since $M \subset \bar{M}$ and $\bar{M}$ is closed by assumption, it follows that $(0, \tilde{w}) \in \bar{M}$. By the definition of the set $\bar{M}$, this implies that there exists some $\bar{w}$ with $\bar{w} \leq \tilde{w}$ and $(0, \bar{w}) \in M$. Hence we have

   $$w^* = \inf_{(0,w)\in M} w \leq \bar{w} \leq \tilde{w} = \liminf_{k\to\infty} w_k,$$

   proving the desired result and showing that $q^*\ w^*$.

2. Consider MC/MC framework and assume $w^* > -\infty$, $\bar{M}$ is convex and $0 \in \text{ri}(D)$, where $D = \{u|$ there exists $w \in \mathbb{R}$ with $(u, w) \in \bar{M}\}$. Here Q^* is the set of optimal solutions of the max crossing problem. Show that
   (a) $(\text{aff}(D))^\perp \subset L_{Q^*}$, where L_S is the lineality space of the set S;
   (b) $R_{Q^*} \subset (\text{aff}(D))^\perp$, where R_S is the recession cone of the set S;
   (c) Show that $\tilde{Q}$ is compact, where $\tilde{Q} = Q^* \cap (\text{aff}(D))^\perp$.

   Solution Please refer to proof of Proposition 4.4.2 in Appendix.

3. Considering MC/MC framework, assuming that $w^* < \infty$, and M is closed and convex, does not contain a halfline of the form $\{(x, w + \alpha)|\alpha \leq 0\}$. Show that

   $$f(x) = \inf\{w|(x, w) \in M\}, \quad x \in \mathbb{R}^n$$

   is closed.

   Solution To show that f is closed, we argue by contradiction. If f is not closed, there exists a vector x and a sequence $\{x_k\}$ that converges to x and is such that

   $$f(x) > \lim_{k\to\infty} f(x_k).$$

   We claim that $\lim_{k\to\infty} f(x_k)$ is finite, i.e., that $\lim_{k\to\infty} f(x_k) > -\infty$. Indeed, by Nonvertical Hyperplane Theorem (Prop. 1.5.8), the epigraph of f is contained in the upper halfspace of a nonvertical hyperplane of $\mathbb{R}^{n+1}$. Since $\{x_k\}$ converges to x, the limit of $f(x_k)$ cannot be equal to $-\infty$. Thus the sequence $(x_k, f(x_k))$, which belongs to M, converges to $(x, \lim_{k\to\infty} f(x_k))$ Therefore, since M is closed, $(x, \lim_{k\to\infty} f(x_k) \in M$. By the definition of f, this implies that $f(x) \leq \lim_{k\to\infty} f(x_k)$, contradicting our earlier hypothesis.

Appendix

Proof of Proposition 4.4.2:

Proof: By Prop. 4.4.1, q^* is finite and Q^* is nonempty. Since q is concave and upper semicontinuous (cf. Prop. 4.1.1), and $Q^* = \{\mu \mid q(\mu) \geq q^*\}$, it follows that Q^* is convex and closed. We will first show that the recession cone R_{Q^*} and the lineality space L_{Q^*} of Q^* are both equal to $\big(\text{aff}(D)\big)^\perp$ [note here that $\text{aff}(D)$ is a subspace since it contains the origin]. The proof of this is based on the generic relation $L_{Q^*} \subset R_{Q^*}$ and the following two relations

$$\big(\text{aff}(D)\big)^\perp \subset L_{Q^*}, \qquad R_{Q^*} \subset \big(\text{aff}(D)\big)^\perp,$$

which we show next.

Let d be a vector in $\big(\text{aff}(D)\big)^\perp$, so that $d'u = 0$ for all $u \in D$. For any vector $\mu \in Q^*$ and any scalar α, we then have

$$q(\mu + \alpha d) = \inf_{(u,w)\in\overline{M}} \{(\mu + \alpha d)'u + w\} = \inf_{(u,w)\in\overline{M}} \{\mu'u + w\} = q(\mu),$$

so that $\mu + \alpha d \in Q^*$. Hence $d \in L_{Q^*}$, and it follows that $\big(\text{aff}(D)\big)^\perp \subset L_{Q^*}$.

Let d be a vector in R_{Q^*}, so that for any $\mu \in Q^*$ and $\alpha \geq 0$,

$$q(\mu + \alpha d) = \inf_{(u,w)\in\overline{M}} \big\{(\mu + \alpha d)'u + w\big\} = q^*.$$

Since $0 \in \text{ri}(D)$, for any $u \in \text{aff}(D)$, there exists a positive scalar γ such that the vectors γu and $-\gamma u$ are in D. By the definition of D, there exist scalars w^+ and w^- such that the pairs $(\gamma u, w^+)$ and $(-\gamma u, w^-)$ are in $\overline{M}$. Using the preceding equation, it follows that for any $\mu \in Q^*$, we have

$$(\mu + \alpha d)'(\gamma u) + w^+ \geq q^*, \qquad \forall\, \alpha \geq 0,$$

$$(\mu + \alpha d)'(-\gamma u) + w^- \geq q^*, \qquad \forall\, \alpha \geq 0.$$

If $d'u \neq 0$, then for sufficiently large $\alpha \geq 0$, one of the preceding two relations will be violated. Thus we must have $d'u = 0$, showing that $d \in \big(\text{aff}(D)\big)^\perp$ and implying that

$$R_{Q^*} \subset \big(\text{aff}(D)\big)^\perp.$$

This relation, together with the generic relation $L_{Q^*} \subset R_{Q^*}$ and the relation $\big(\text{aff}(D)\big)^\perp \subset L_{Q^*}$ proved earlier, shows that

$$\big(\text{aff}(D)\big)^\perp \subset L_{Q^*} \subset R_{Q^*} \subset \big(\text{aff}(D)\big)^\perp.$$

Therefore

$$L_{Q^*} = R_{Q^*} = \big(\text{aff}(D)\big)^\perp.$$

We now use the decomposition result of Prop. 1.4.4, to assert that

$$Q^* = L_{Q^*} + (Q^* \cap L_{Q^*}^\perp).$$

Since $L_{Q^*} = \big(\text{aff}(D)\big)^\perp$, we obtain

$$Q^* = \big(\text{aff}(D)\big)^\perp + \tilde{Q},$$

where $\tilde{Q} = Q^* \cap \text{aff}(D)$. Furthermore, by Prop. 1.4.2(c), we have

$$R_{\tilde{Q}} = R_{Q^*} \cap R_{\text{aff}(D)}.$$

Since $R_{Q^*} = \big(\text{aff}(D)\big)^\perp$, as shown earlier, and $R_{\text{aff}(D)} = \text{aff}(D)$, the recession cone $R_{\tilde{Q}}$ consists of the zero vector only, implying that the set $\tilde{Q}$ is compact.

From the formula $Q^* = \big(\text{aff}(D)\big)^\perp + \tilde{Q}$, it follows that Q^* is compact if and only if $\big(\text{aff}(D)\big)^\perp = \{0\}$, or equivalently $\text{aff}(D) = \Re^n$. Since 0 is a relative interior point of D by assumption, this is equivalent to 0 being an interior point of D. **Q.E.D.**